UCRL- 91877
PREPRINT

SPLINES REVISITED

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This paper was prepared for presentation at
NASIG'84 -- Argonne National Laboratory
September 18-19, 1984

September 1984

Lawrence Livermore National Laboratory

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SPLINES REVISITED*

This report is a reproduction of slides used in a talk given 18 September 1984 at NASIG'84.

ABSTRACT

The Fowler-Wilson spline, long used in the CAM system APT, and the beta-spline, recently introduced for computed-aided curve and surface design, are defined and compared. These "splines" are also compared with the more familiar cubic B-splines.

*This work was performed under the auspices of the U.S. Department of Energy by Lawrence Livermore National Laboratory under contract No. W-7405-Eng-48.

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Splines Revisited

or

Just What Is This Alphabet Soup?

Outline

1. History of splines
2. Cubic splines
3. General piecewise polynomial functions; B-splines
4. Parametric splines; B-spline curves.
5. Interactive design of curves
6. β-splines
7. The Fowler-Wilson spline
8. Connections between F-W and other splines

A Brief History of Splines

Reference: L. Schumaker, *Spline Functions: Basic Theory*, Wiley, 1981.

- Mathematical model for the mechanical spline.
  [Ahlberg; Nilson; Schoenberg; Walsh - early 1960's]

- General piecewise polynomial functions; B-splines.
  [deBoor; Cox - early 1970's]

- Parametric splines; B-spline curves.
  [various systems for designing curves/surfaces]

- Generalizations: splines under tension; ν-splines;
  L-splines; . . . ; β-splines [Barsky - 1981]

- Nonlinear splines.
  [Fowler&Wilson - 1963; Birkhoff,Burchard,Thomas - 1965;
  Lee&Forsythe - 1973; Mehlum - 1974].

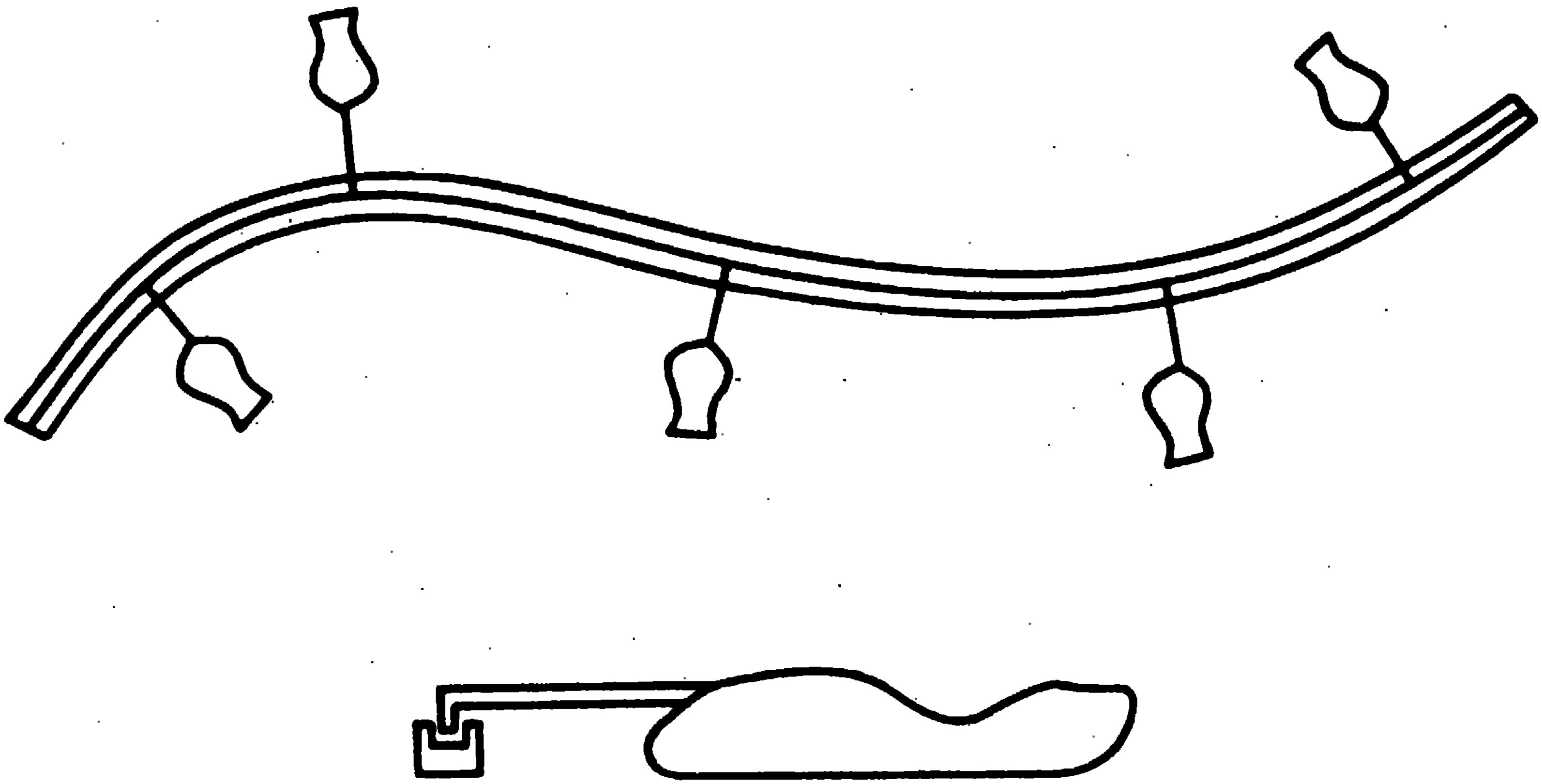


Figure 2. The mechanical spline.

[From Schumaker, op. cit., p. 6.]

Cubic Splines

Given: $x_1 < x_2 < \dots < x_n$; $y_i = f(x_i)$, $i=1,\dots,n$.

- $s(x)$ is a cubic polynomial in each subinterval $[x_i, x_{i+1}]$.
- $s(x)$, $s'(x)$, $s''(x)$ are all continuous on $[x_1, x_n]$.

Properties:

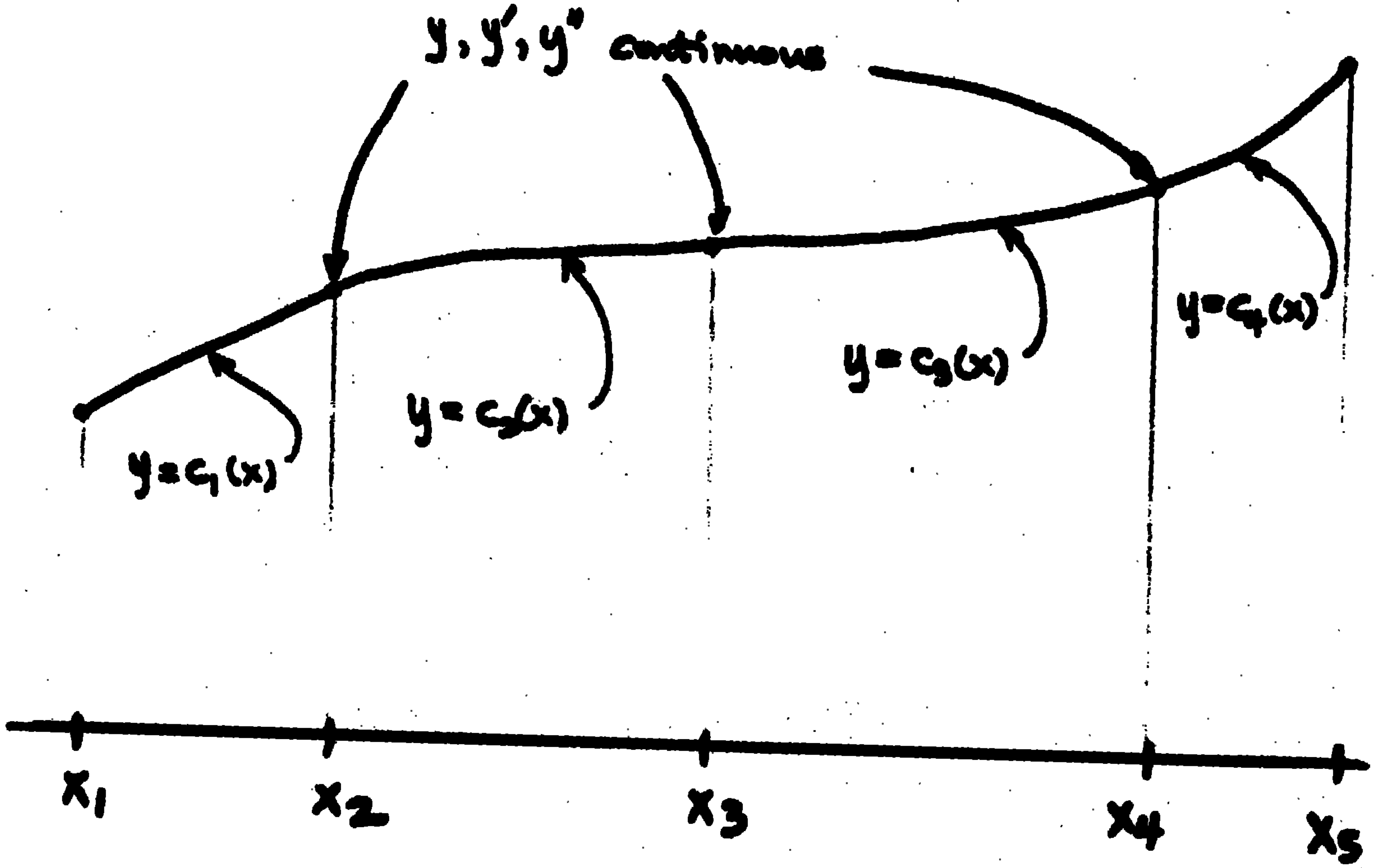
- Two degrees of freedom (BC) for interpolation problem:

$$s(x_i) = y_i, \quad i = 1,\dots,n.$$

- "Natural" spline, with $s''(x_1) = s''(x_n) = 0$, minimizes the "energy" integral, $\int [f''(x)]^2\, dx$.

- Solve diagonally-dominant tridiagonal linear system.

A Typical Cubic Spline



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General Piecewise Polynomial Functions

- $s(x)$ is a polynomial of order k (degree $\leq k-1$) in $[x_i, x_{i+1}]$.

- Arbitrary continuity requirements at $x_2, \ldots, x_{n-1}$ [maximum $(k-2)$nd derivative, else "not a knot"] .

- B-splines provide a stable basis for computation (B-spline package):

  1. $s(x) = \sum c_j B_j(x)$.

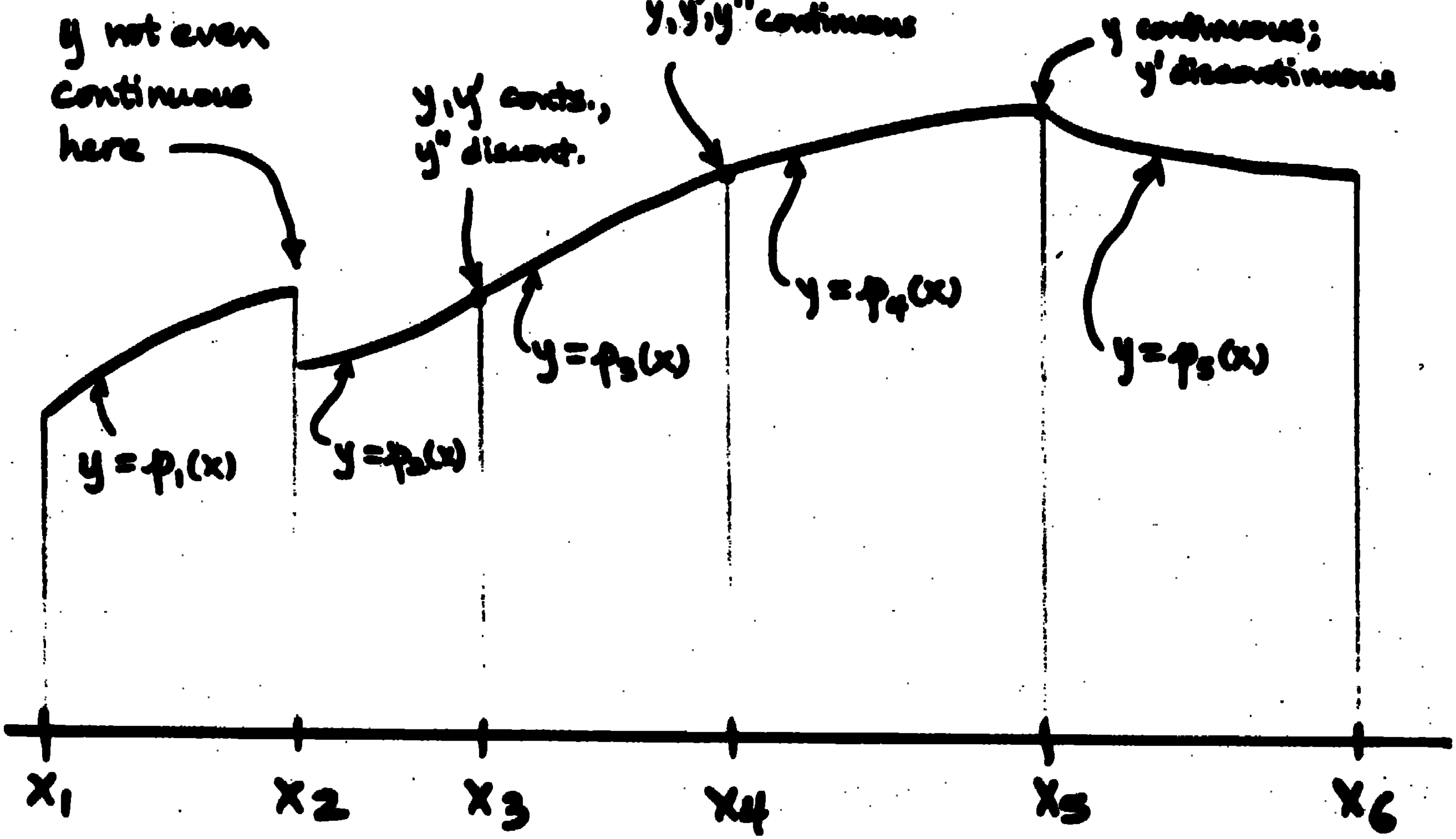
  2. $B_j(x) \geq 0$; $B_j(x) = 0$ except on k intervals.

  3. $\sum B_j(x) = 1$.

- Solve (diagonally-dominant) banded linear system.

A General Piecewise Polynomial Function

($k \geq 4$)



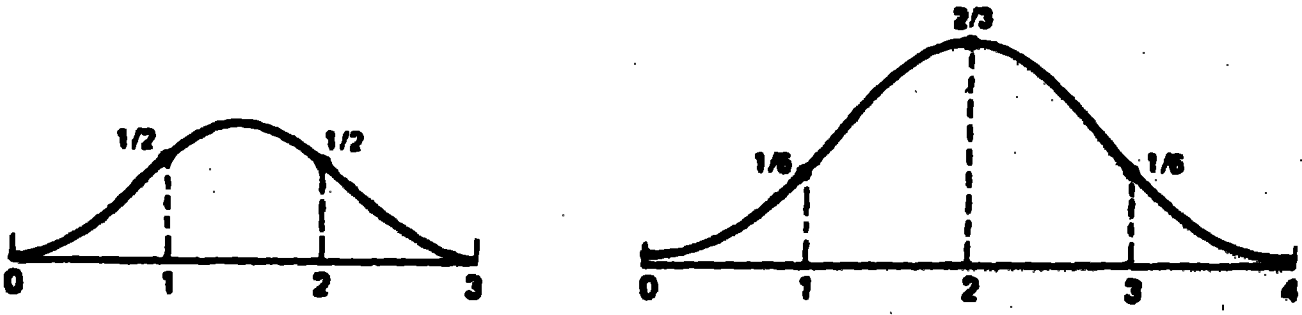


Figure 10. The B-splines N^3 and N^4.

[From Schumaker, op. cit., p. 136.]

B-SPLINES **123**

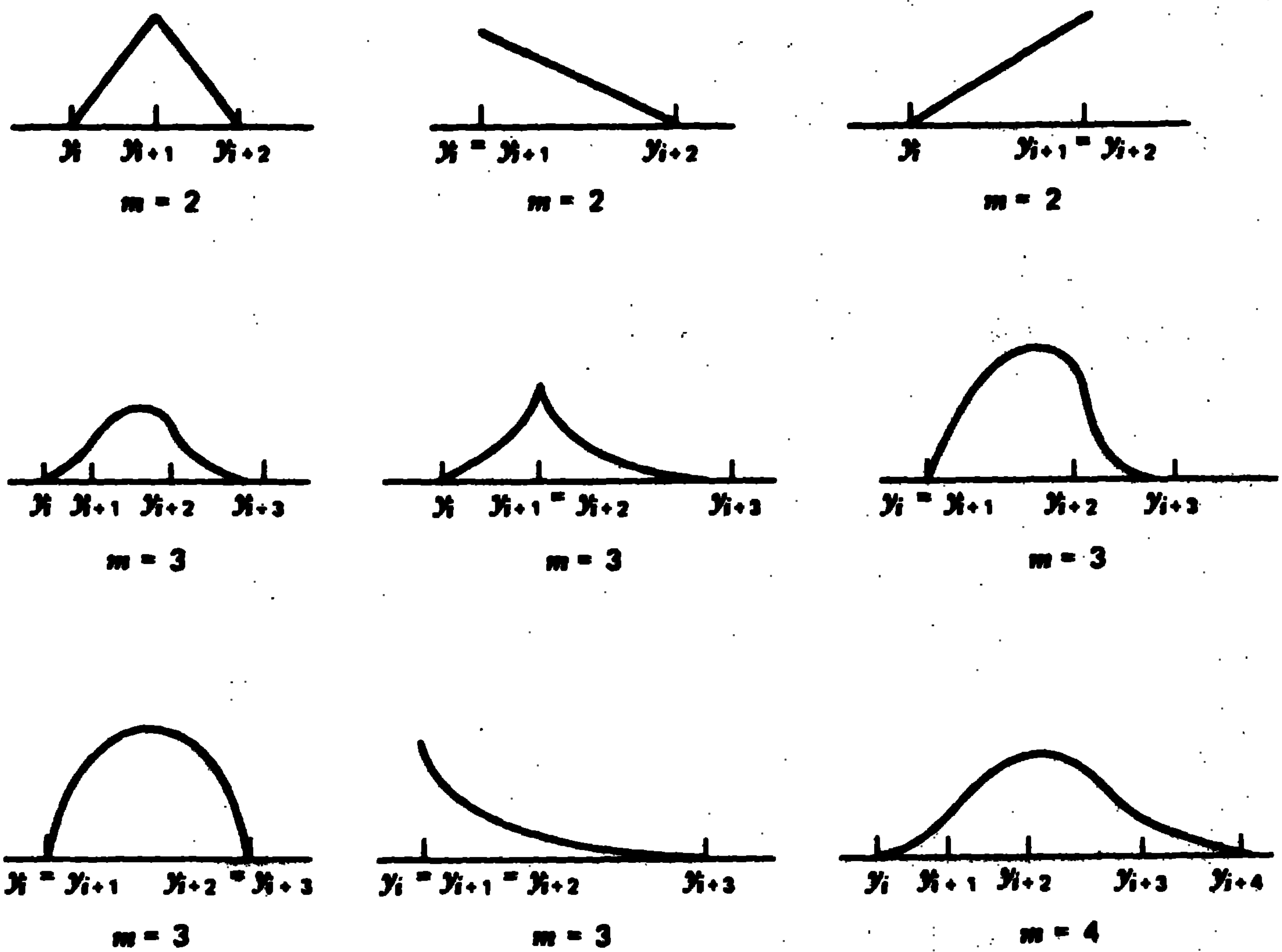


Figure 7. Shapes of some B-splines.

[From Schumaker, op. cit., p. 123.]

NASIG'84-4b

Parametric Splines

Given: $P_1, P_2, \ldots, P_n$; $P_i = (x_i, y_i)$, $i=1,\ldots,n$.

- View as a collection of points in the plane through which a smooth curve is to be drawn. [No implied functional dependence, $y = f(x)$.]

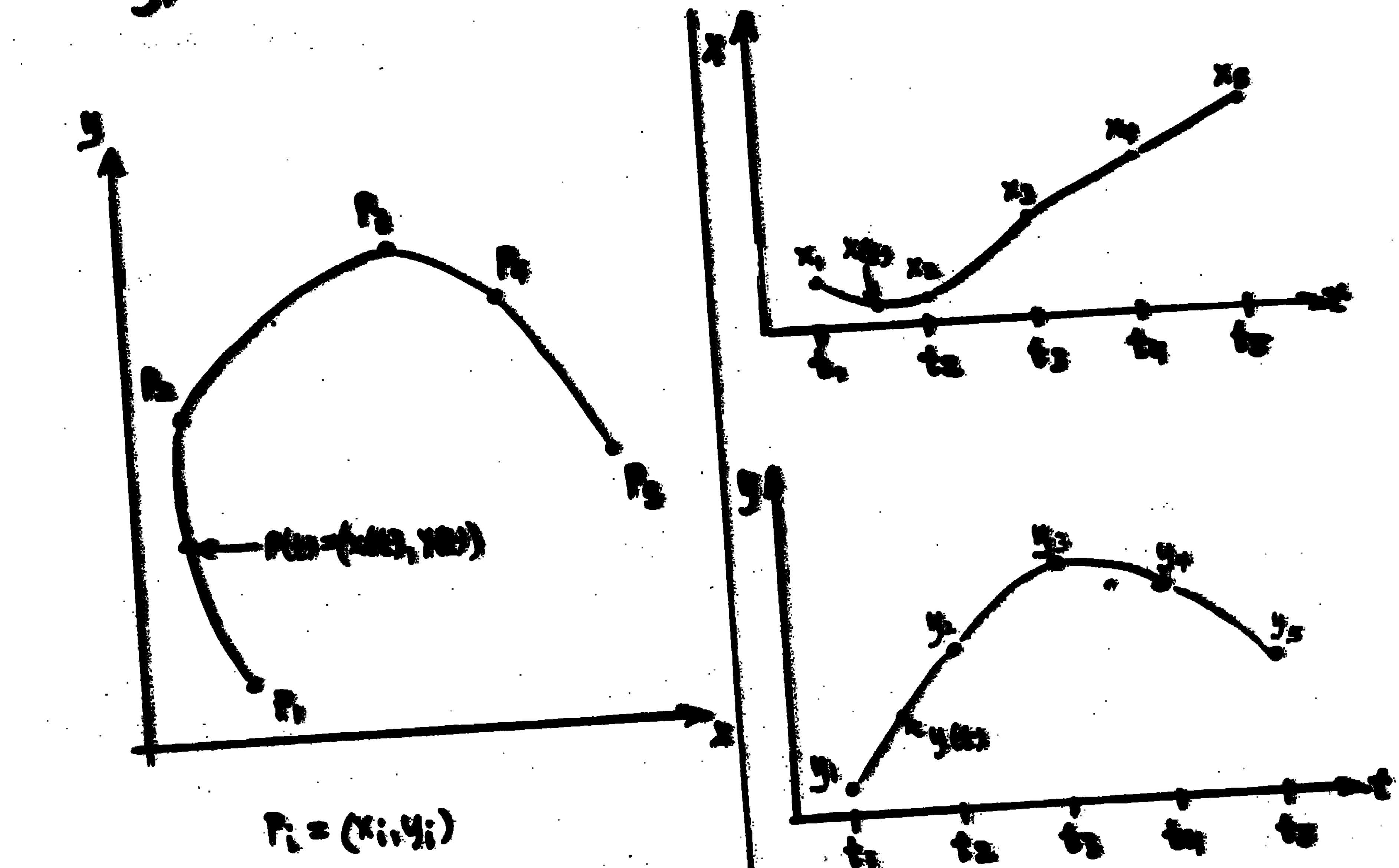
- Introduce a parametrization via $t_1 < t_2 < \ldots < t_n$.

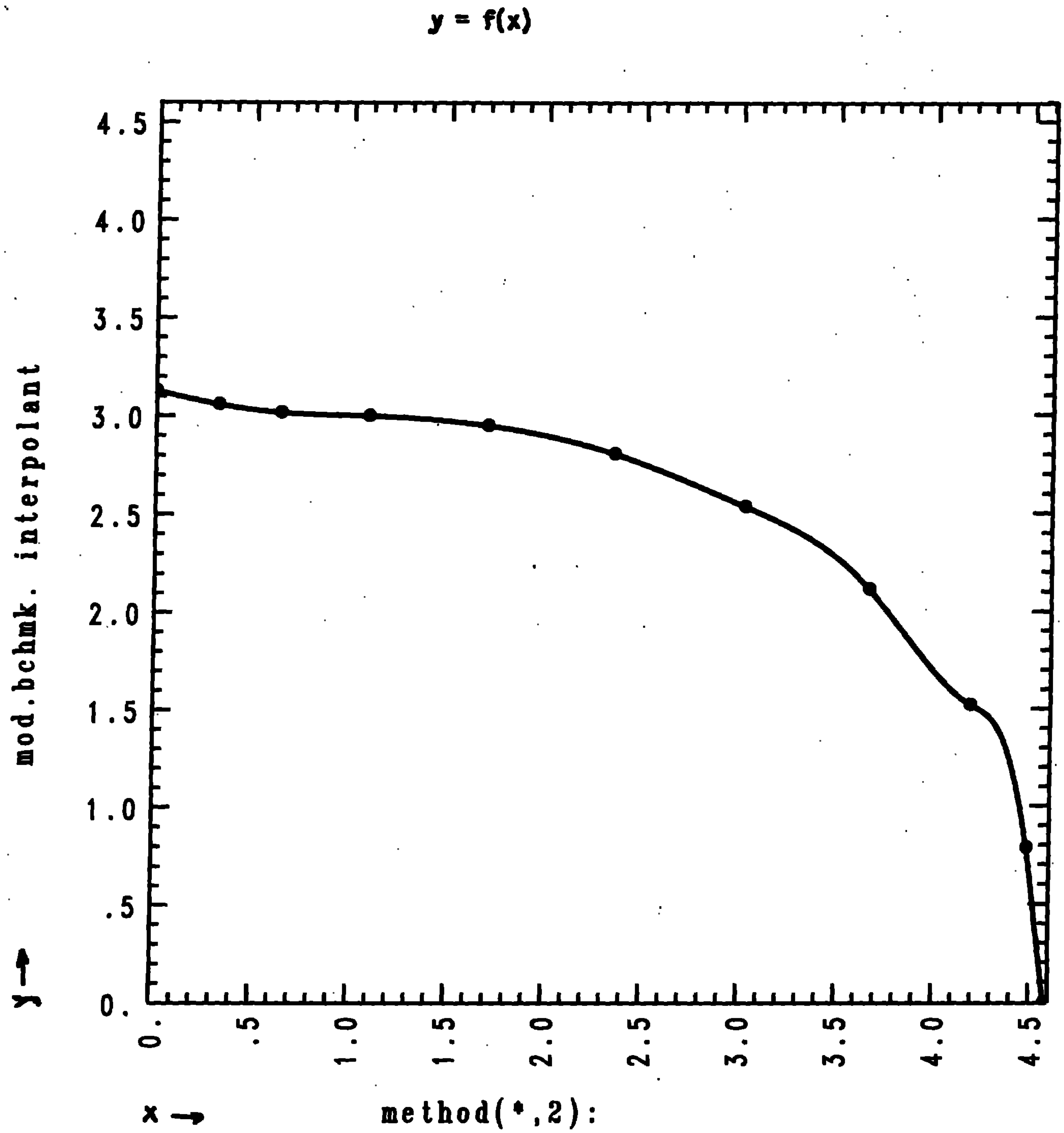
- Define curve via two spline functions in parameter space:

$$x = f(t); \quad f(t_i) = x_i,\ i=1,\ldots,n ;$$
$$y = g(t); \quad g(t_i) = y_i,\ i=1,\ldots,n .$$

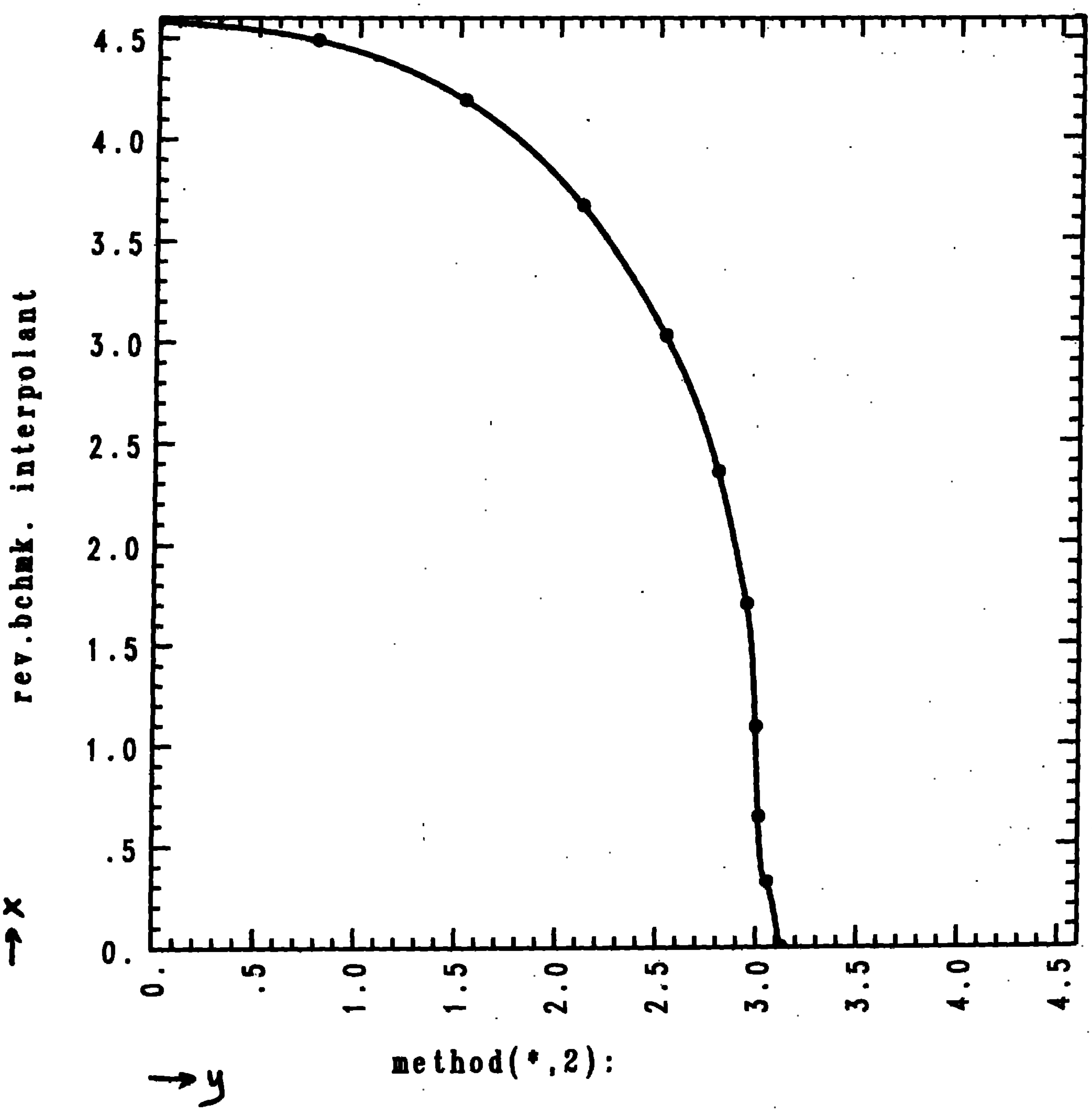
- B-spline curve if f and g are B-splines.
  [Typically, f and g are quadratic or cubic splines.]

A Typical Parametric Spline




y = f(x)
mod.bchmk. interpolant
y →
x →
method(*,2):
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
.5
0.
0.
.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5

$x = \hat{f}(y)$



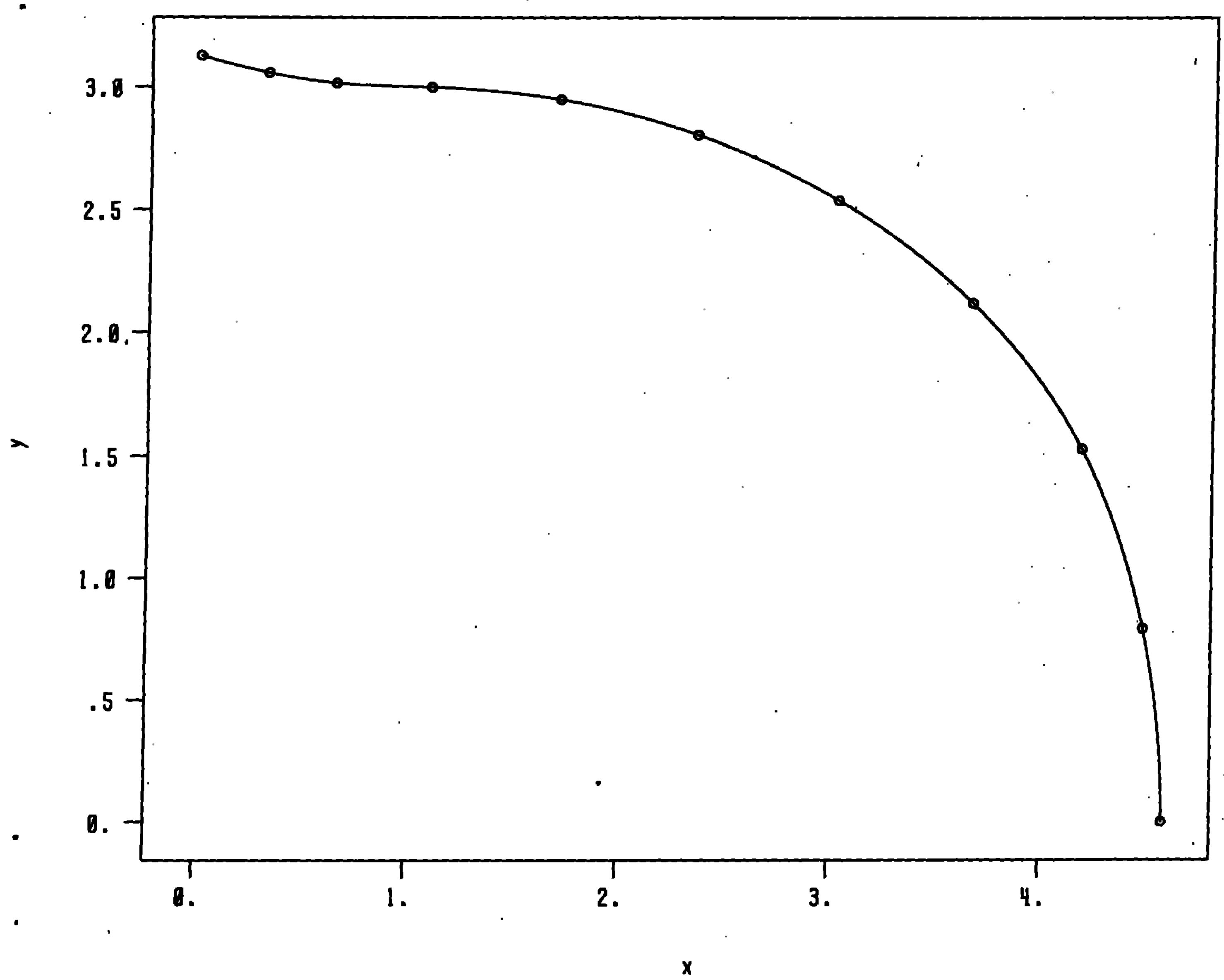
XSIG'34-5c

Parametric cubic spline plot code -- version 9

LLL benchmark test (selected) -- x,y -- default end conditions

ibeg = 0, vbeg = 0. iend = 0, vend = 0. npar = 1

Curve and Data Points



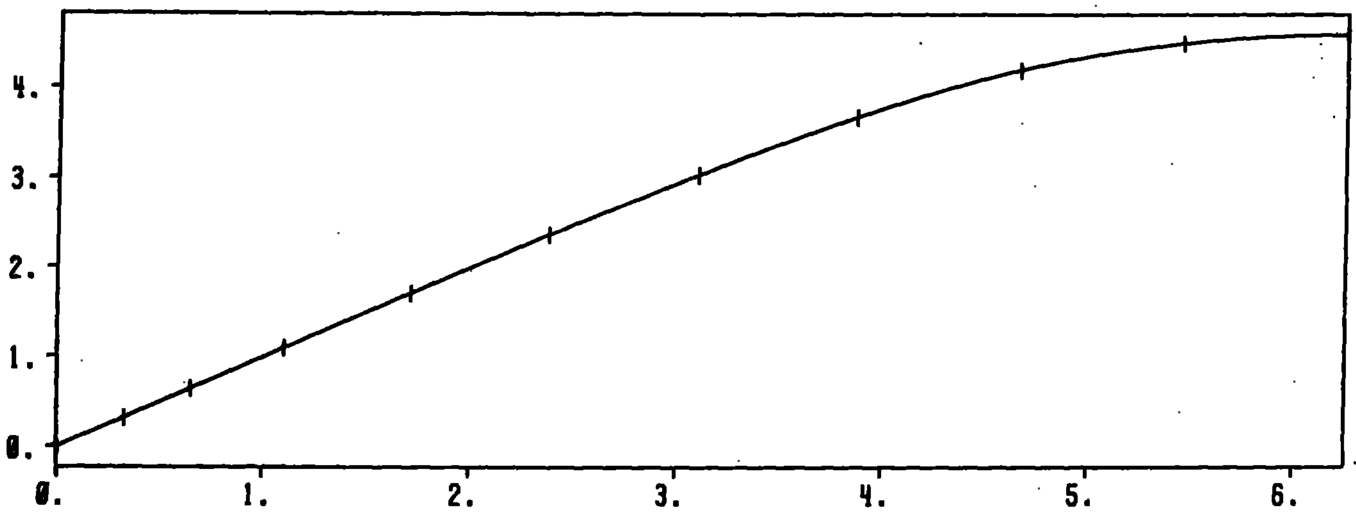
HASIG'84-5d

Parametric cubic spline plot code -- version 9

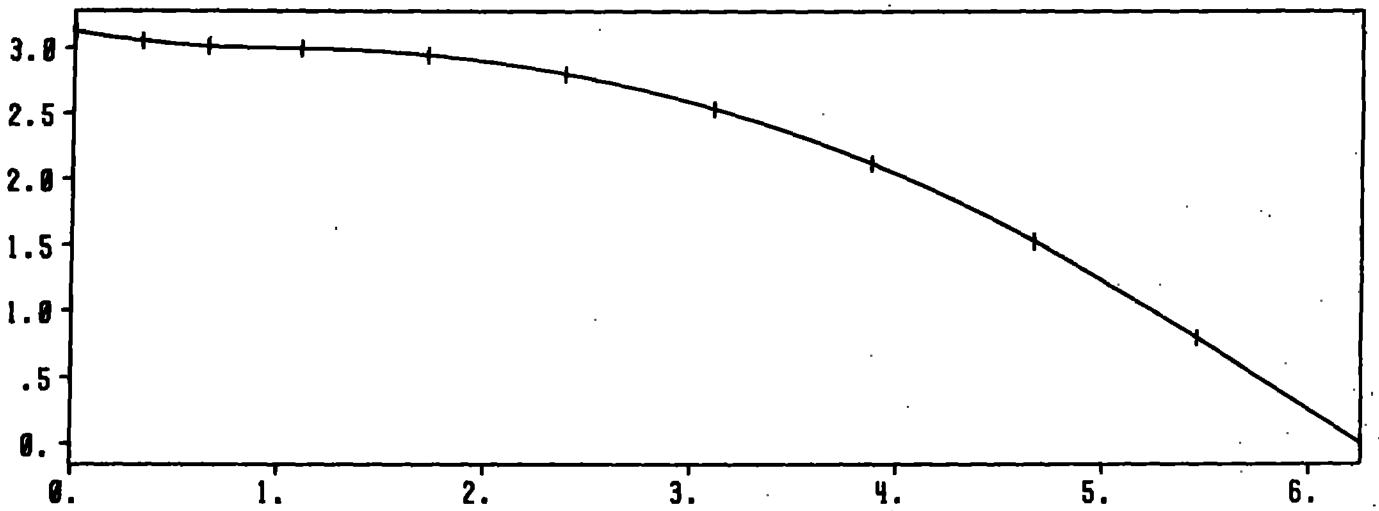
LLL benchmark test (selected) -- x,y -- default end conditions

ibeg = 0, vbeg = 0. iend = 0, vend = 0. npar = 1

x component vs parameter t



y component vs parameter t



NASTG'34-5e

Interactive Design of Curves

- Change of motivation: design of free-form curves (rather than interpolation of "hard" data).

- Control polygon $Q = Q_1 Q_2 \ldots Q_m$:

  $Q_j = (a_j, b_j)$, where

  $$x = f(t) = \sum a_j B_j(t) ;$$
  $$y = g(t) = \sum b_j B_j(t) .$$

- Curve $P(t) = (f(t), g(t))$ mimics the shape of Q .

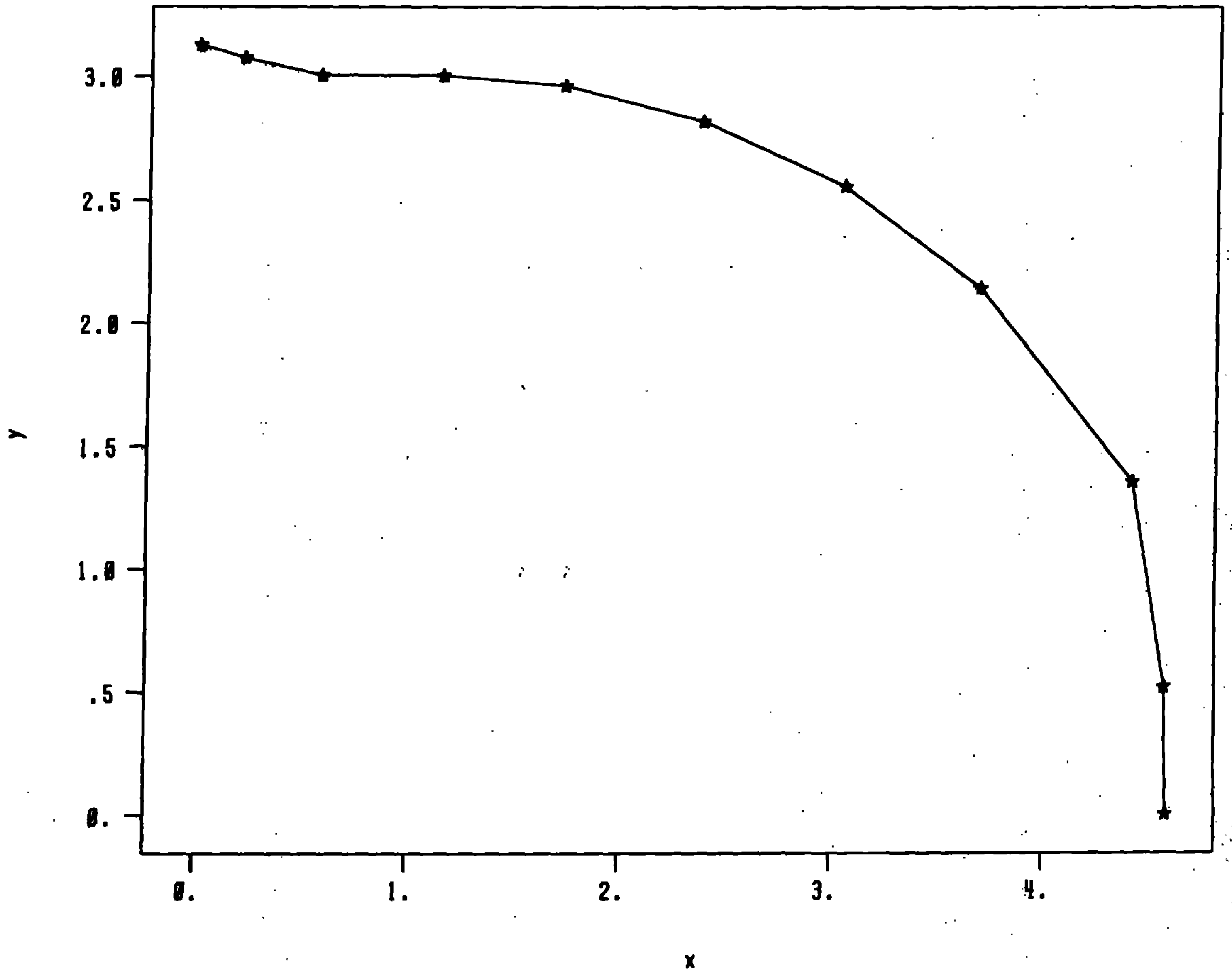
- Extends naturally to more independent variables.

Parametric cubic spline plot code -- version 12

LLL benchmark test (selected) -- x,y -- default end conditions

ibeg = 0, vbeg = 0.   iend = 0, vend = 0.   npar = 1

Control Polygon



β-Splines

- Generalization of the cubic B-spline curve, introduced to obtain more local control over shape of curve segments.

- Parametric continuity:

$$(0)\ P^{(0)}(t_i-) = P^{(0)}(t_i+)\ ,$$
$$(1)\ P^{(1)}(t_i-) = P^{(1)}(t_i+)\ ,$$
$$(2)\ P^{(2)}(t_i-) = P^{(2)}(t_i+)\ ,\ i=2,\ldots,n-1\ ,$$

where $P^{(k)}(t) = (f^{(k)}(t), g^{(k)}(t))$.

- Geometric continuity [Barsky]:

Curve, unit tangent, and curvature continuous on $[t_1, t_n]$.

$$(0)\ P^{(0)}(t_i-) = P^{(0)}(t_i+)\ ,$$
$$(1)\ \beta_{1i}\, P^{(1)}(t_i-) = P^{(1)}(t_i+)\ ,$$
$$(2)\ \beta_{1i}^{\ 2}\, P^{(2)}(t_i-) + \beta_{2i}\, P^{(1)}(t_i-) = P^{(2)}(t_i+)\ .$$

[2(n-2) extra parameters β_{1i} and β_{2i} , i=2,...,n-1 .]

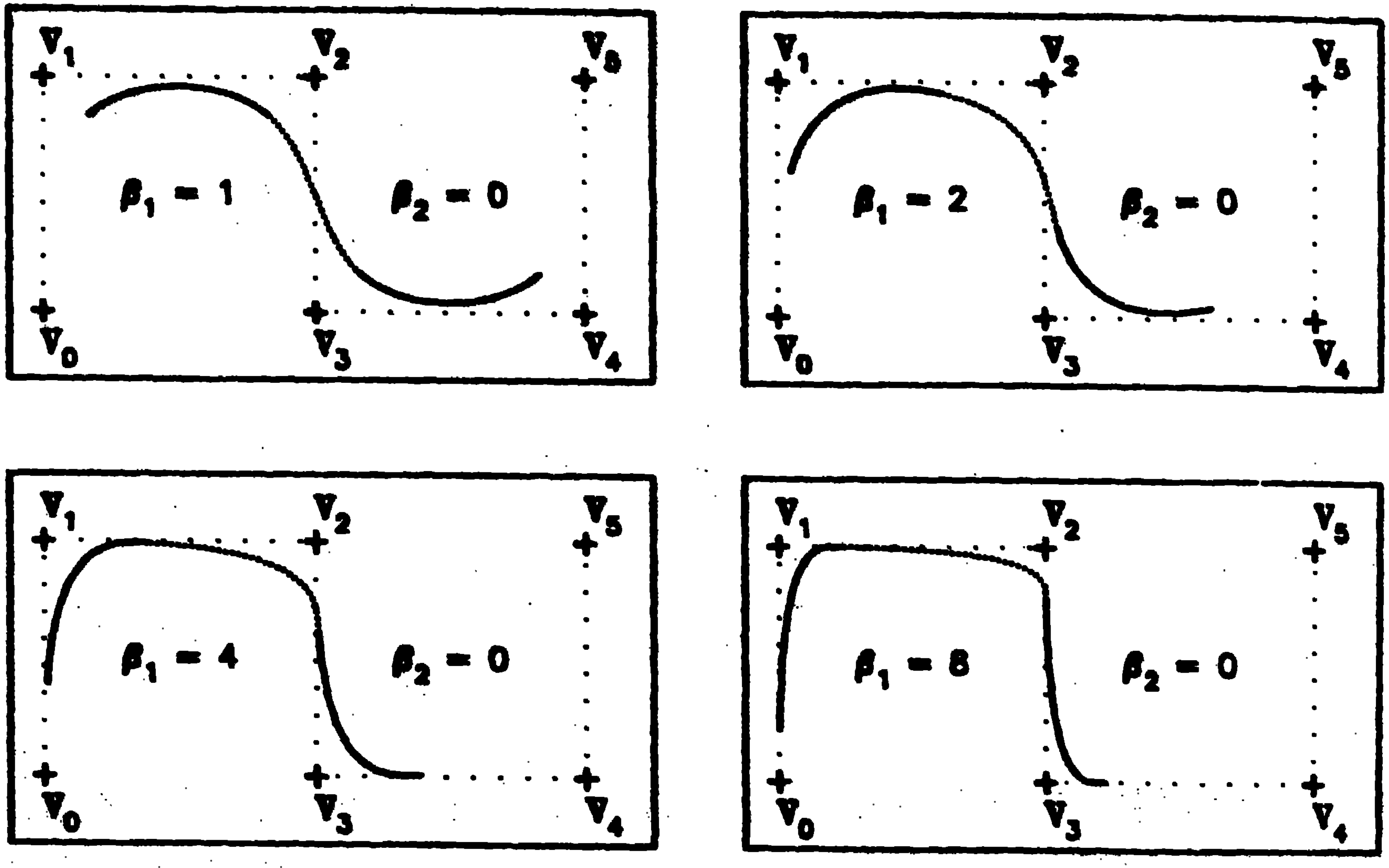


Figure 115. This sequence of curves illustrates the effect of increasing β_1 on a uniformly-shaped Beta-spline.

[From R.H.Bartels, J.C.Beatty, and B.A.Barsky, An Introduction to the Use of Splines in Computer Graphics, University of Waterloo Report CS-83-09, August 1983, p. 160.]

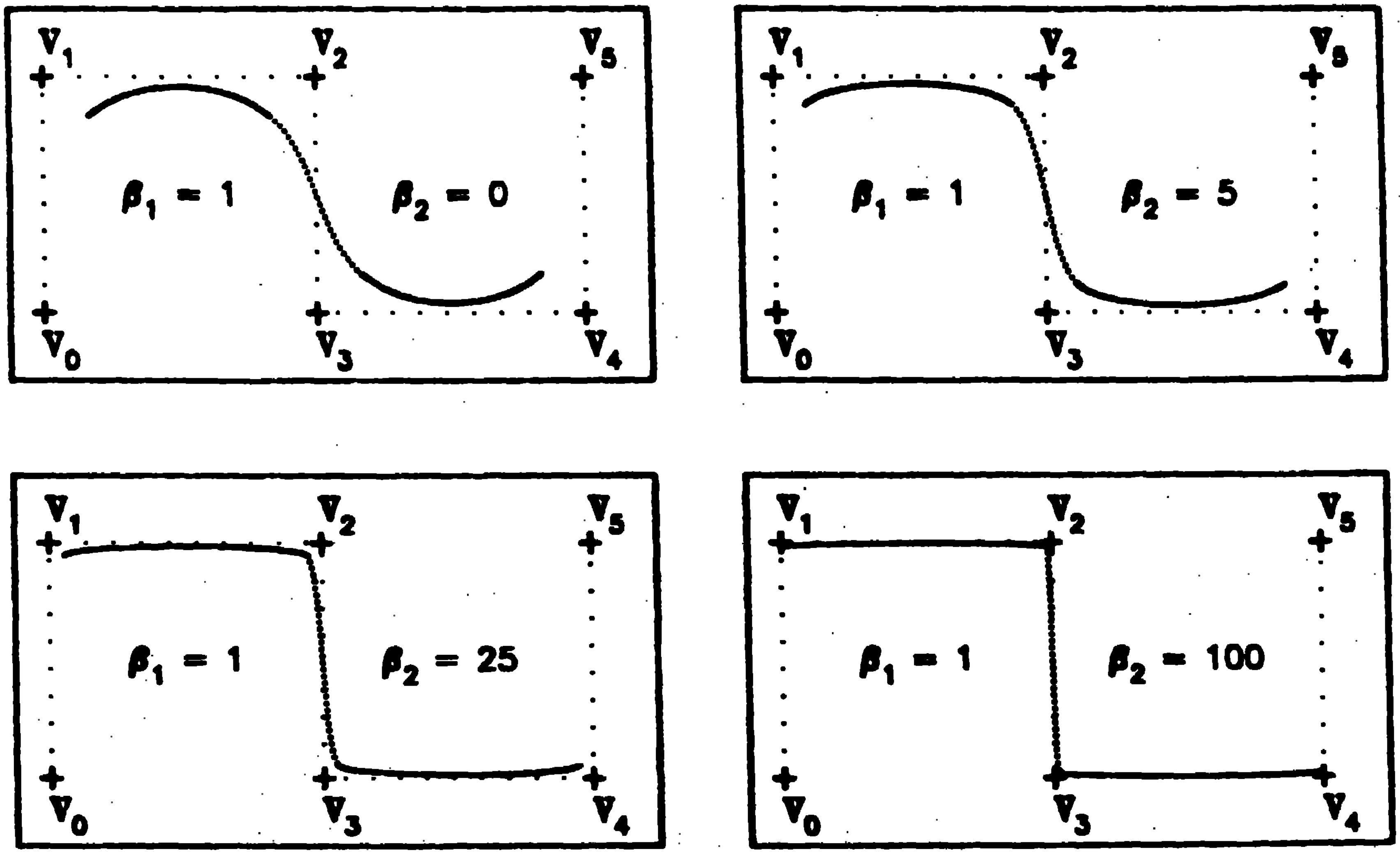


Figure 117. **This sequence of curves illustrates the effect of increasing β_2 on a uniformly-shaped Beta-spline.**

[From Bartels, Beatty, and Barsky, op. cit., p. 161.]

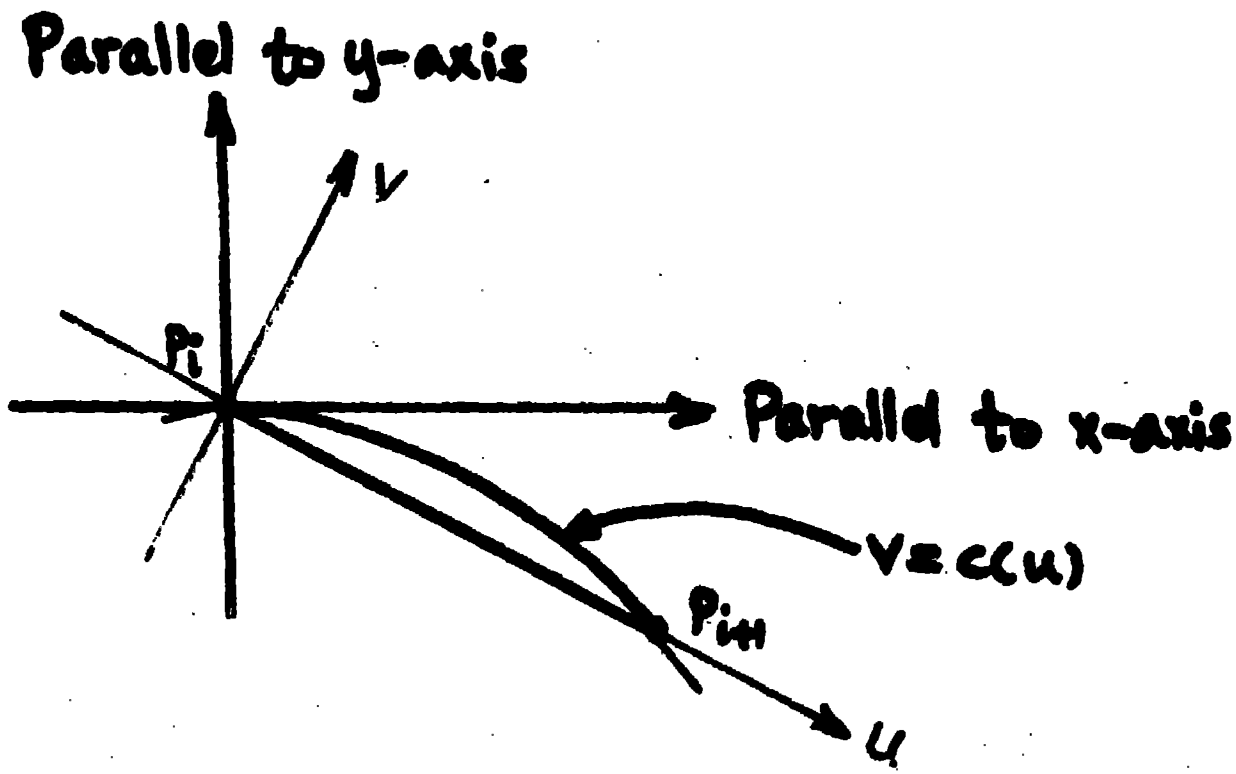
The Fowler-Wilson Spline

- Generalizes "ordinary" cubic spline to points in plane in a different way. (Back to interpolation of "hard" data.)

- Definition:
  (1) Use local coordinate system for each curve segment; curve position is cubic in local coordinates.
  (2) Require continuous tangent direction and curvature at interior points. [Solve tridiagonal *non*linear system.]

- History:
  - First described in 1963 Oak Ridge report by A.H.Fowler and C.W.Wilson (with iterative algorithm).
  - Has been in use in APT system since early 1960's.
  - W.R.Melvin studied existence and uniqueness properties, gave much better Newton iteration in 1982 LANL report.

F-W Spline Local Coordinate System

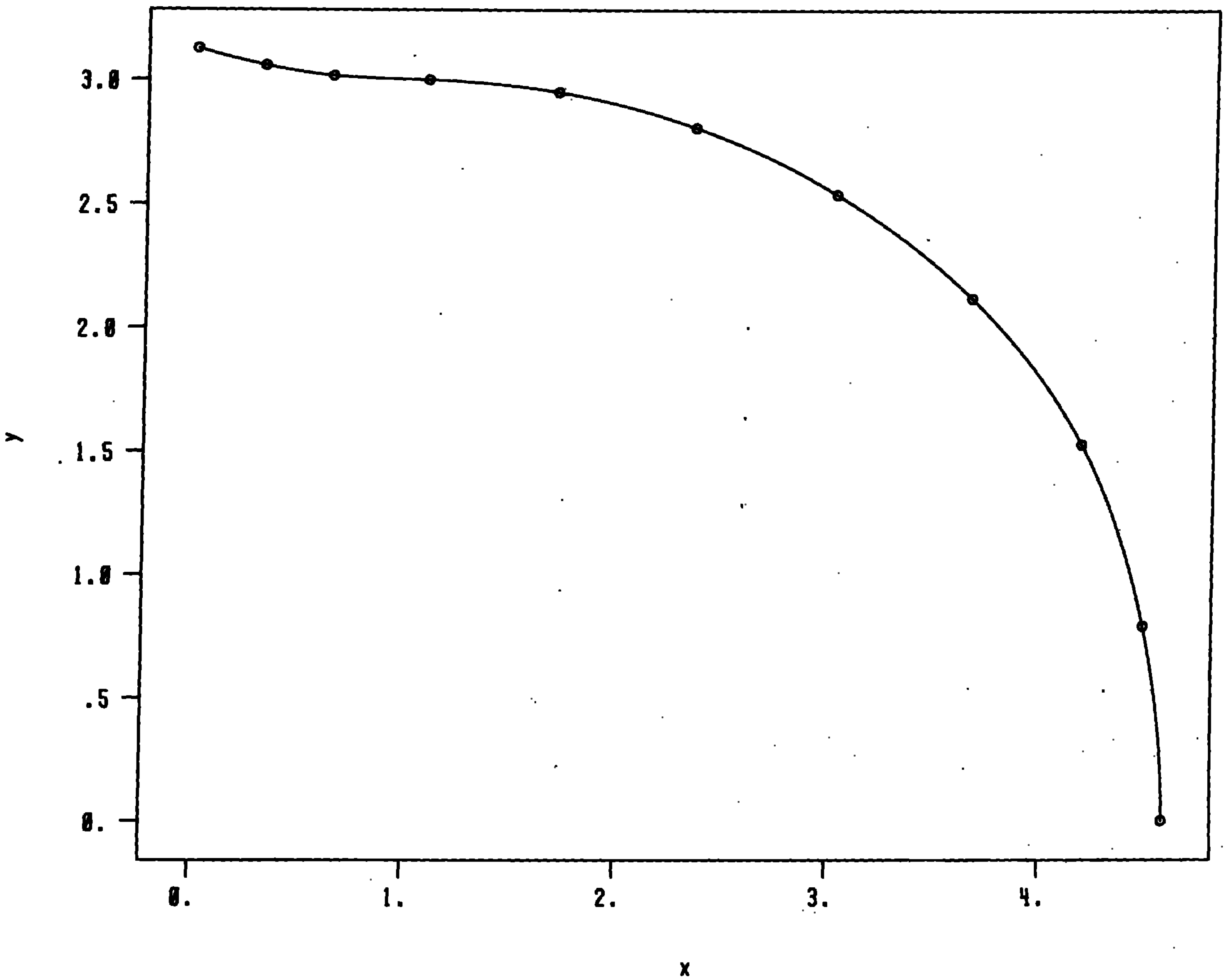


Fowler-Wilson spline plot code -- version 16

LLL benchmark test (selected) -- x,y -- default end conditions

ibeg = 0. entrya = -2.52679e-01   iend = 0. exita = -1.58496e+00   eps = 1.0e-03

Curve and Data Points

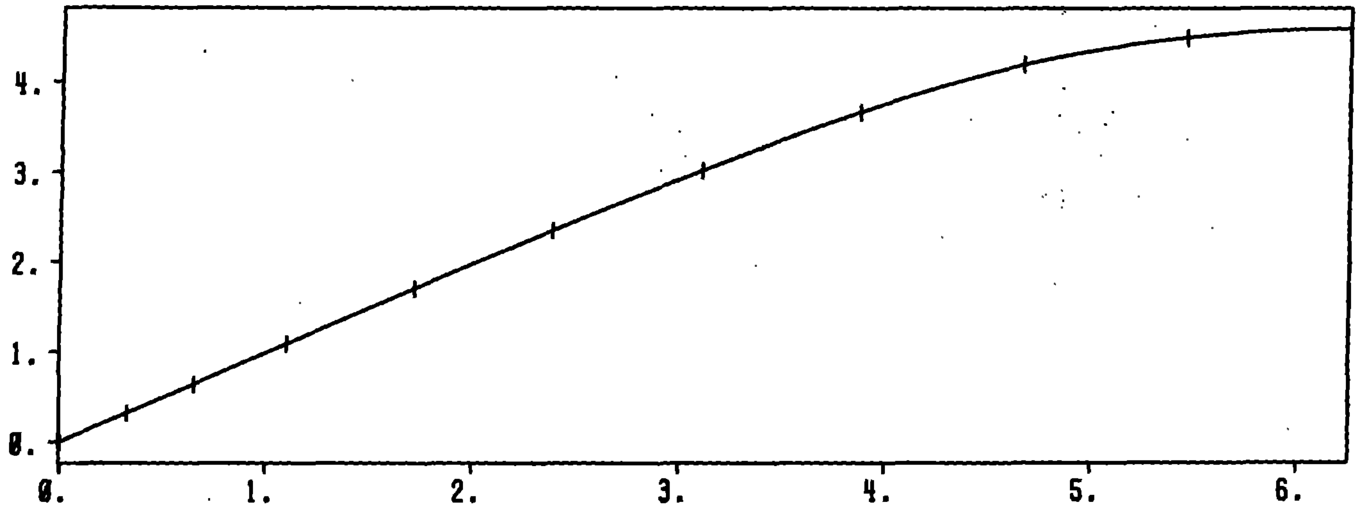


Fowler-Wilson spline plot code -- version 16

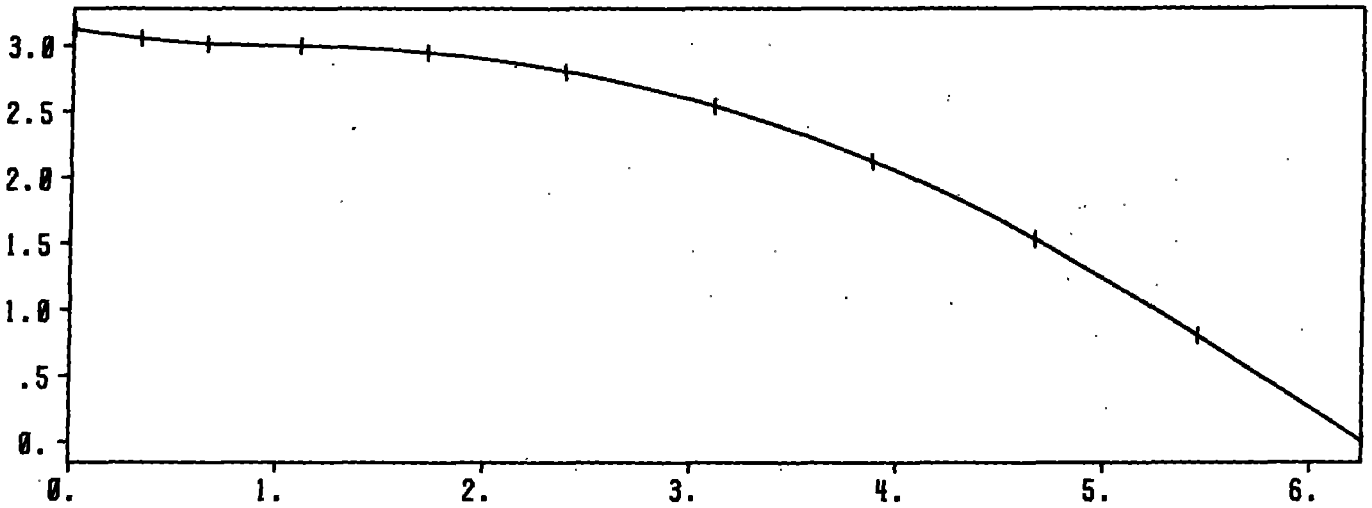
LLL benchmark test (selected) -- x,y -- default end conditions

ibeg = 0, entrya =-2.52679e-01 iend = 0, exita =-1.58496e+00 eps = 1.0e-03

x component vs parameter t



y component vs parameter t



Connections between F-W and Other Splines

- F-W spline is a parametric piecewise cubic curve;
  it is *not* a parametric cubic spline.
  [Component functions not even C^1.]

- Can reparametrize to be C^1 but not C^2 (in general).
  [Representable as a B-spline curve with double knots.]

- The F-W spline is a β-spline.